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EQUIVALENCE OF STABLE MAPPINGS BETWEEN TWO-DIMENSIONAL MANIFOLDS

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1. Introduction

In this paper, we study stable C^∞ mappings between two-dimensional manifolds. For any stable map $f: M \to N$ we define stratifications of M and N (partitions of M and N into submanifolds, called strata) such that f maps each stratum in M diffeomorphically onto a stratum in N. Suppose f and g are stable maps from M to N, and there exists a homeomorphism $h: M \to M$ inducing a one-to-one correspondence between the stratifications defined on M by f and g such that $(g \circ h)(S) = f(S)$ for each stratum S defined by f. Then there exists a C^∞ diffeomorphism $h': M \to M$ such that $g \circ h' = f$. The above is essentially Theorem 4.1, which is the principle result of this paper and is stated and proved in § 4. The aforementioned stratifications are defined in § 2. In § 3, we give examples illustrating some ways in which Theorem 4.1 cannot be strengthened. In the rest of this section, we make definitions and state known results which will be used later. The material of this paper, except for the proof of Theorem 4.1, is contained in the author's thesis [20].

In [18], C^r mappings ($r \geq 3$) from an open set $U \subseteq \boldsymbol{R}^2$ into $\boldsymbol{R}^2$ were studied by Whitney. Let f be such a mapping, and J its Jacobian matrix. If $\det J(p) \neq 0$, then $p \in U$ is said to be a *regular point* of f; otherwise, p is a *singular point*. Whitney calls p a *good point* for f if p is regular or if $\operatorname{grad}(\det J(p)) \neq 0$. If f is good, that is, each point of U is good for f, then the singular set $\det J = 0$ is a 1-manifold (by the implicit function theorem). Suppose p is a singular point of a good map f with $\varphi(t)$ a regular C^2 parameterization of the singular curve through p. Whitney calls p a *fold point* of f if $d(f \circ \varphi)/\mathrm{d}t \neq 0$ at p, and a *cusp point* of f if $d(f \circ \varphi)/dt = 0$ and $d^2(f \circ \varphi)/dt^2 \neq 0$ at p. The definitions of fold and cusp points are independent of the parameterization φ. Cusp points are necessarily isolated. Thus the set F of fold points of f is a 1-manifold; the connected components of F are called *fold curves*. A point p is an *excellent point* of a good map f if it is either regular or else a fold or a cusp point, and f is excellent if each point of U is excellent for f.

If p is a regular point of f, then, by the inverse mapping theorem, C^r coordinate systems (x, y) and (u, v) exist around p and $f(p)$ respectively such that f takes the form $u = x$, $v = y$.

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Whitney showed, in part C of [18], that:

(1.1.a) if p is a fold point, then C^{r-3} coordinate systems (x, y) and (u, v) exist around p and $f(p)$ respectively such that f takes the form $u = x^2$, $v = y$;

(1.1.b) if p is a cusp point, and $r \geq 12$, then C^k coordinate systems exist around p and $f(p)$, where $k = \frac{1}{2}(r - 5)$ if r is odd and $k = \frac{1}{2}(r - 6)$ if r is even, such that f takes the form $u = xy - x^3$, $v = y$.

For f a C^∞ mapping, Malgrange [6, p. 79] gives a relatively easy proof of (1.1) using his preparation theorem.

Whitney, in part B of [18], proved that:

(1.2) the excellent C^r maps are dense among all C^r maps from U to $\boldsymbol{R}^2$ in the fine (Whitney) C^r topology.

Let $C^\infty(M, N)$ denote the set of C^∞ mappings from the manifold M to the manifold N, and $C^\infty(M)$ the set of C^∞ real-valued functions on M.

Two mappings $f, g \in C^\infty(M, N)$ are said to be *equivalent* if there exist diffeomorphisms $h \in C^\infty(M, M)$ and $k \in C^\infty(N, N)$ such that $k \circ f = g \circ h$.

A mapping $f \in C^\infty(M, N)$ is said to be *stable* if there is a neighborhood W of f in the fine C^∞ topology such that all $g \in W$ are equivalent to f.

Whitney [19, p. 301] defined and briefly discussed stability for mappings between Euclidean spaces. The first extensive study of stability was Levine's [4]. Haefliger [3], and later Levine [5], studied the relationship between the types of stable mappings possible from one manifold to another and the topologies of the manifolds (but only in very restricted dimensions). Many fundamental results about stable mappings have been proven by Mather [7], $\cdots$, [14]. His work has been surveyed by Arnold [1], Cartan [2], and Wall [17]. These surveys contain extensive bibliographies.

If $f \in C^\infty(M, N)$, f is said to be *excellent* if for each $x \in \Sigma$ there are coordinate systems about x and $f(x)$ giving f the form (1.1.a) or (1.1.b).

Proposition 1.3. *Suppose M and N are two-dimensional manifolds, and $f \in C^\infty(M, N)$ is proper. Then f is stable if and only if the following two conditions are satisfied:*

(WI) *f is excellent,*

(WII) *the images of fold curves intersect only pairwise and transversally (that is, if x and y are fold points such that $f(x) = f(y)$, and F denotes the* 1-*manifold of all fold points, then $Tf(T_xF)$ and $Tf(T_yF)$ have regular intersection in $T_{f(x)}N$), whereas images of cusps do not intersect with images of folds or other cusps.*

For a proof of this well-known result, see Wilson [20].

2. Stratifications

In this section, M and N denote two-dimensional, Hausdorff, second countable, C^∞ manifolds, and f and g denote proper, stable C^∞ mappings from M to N. Recall that f and g are equivalent if there are diffeomorphisms $h \in C^\infty(M, M)$ and $k \in C^\infty(N, N)$ such that $k \circ f = g \circ h$. Theorems 2.1 and 4.1 will characterize the equivalence of stable mappings. This charactrization is obtained in terms of certain natural "stratifications" $\mathscr{S}_1, \mathscr{S}_2$ and $\mathscr{S}_3$ (collections of subsets, called strata, of M and N) defined by the mappings f and g, and it involves the notion of a "simple substratification".

First we define the stratifications $\mathscr{S}_1, \mathscr{S}_2$ and $\mathscr{S}_3$ for the map f (cf. Figures 1 through 6 for examples of these stratifications).

Let $R1$ be the set of all regular points of f. Then $R1$ is an open subset of M. Let $F1$ be the set of all fold points. Then $F1$ is an embedded one-dimensional submanifold of M (which is not necessarily connected). Let $C1$ be the set of all cusp points. Then $C1$ is a discrete set. The set $\Sigma = F1 \cup C1$ of all singular points is an embedded one-dimensional submanifold of M, and is a closed subset. Each connected component of Σ is called a *generalized fold curve* of f, and is either a (smooth) Jordan curve or an infinite arc in M. Let $S_{1,x}$ denote the connected component of $R1$, $F1$ or $C1$ containing x. Then $\mathscr{S}_1 := \mathscr{S}_1(f) := \{S_{1,x} : x \in M\}$.

The set of regular values $R2 = N - f(\Sigma)$ is an open subset of N. The set of fold values $F2 = \{y \in N : \#(f^{-1}(y) \cap F1) = 1\}$ is an embedded one-dimensional submanifold of N. The set of double fold values $FF2 = \{y \in N : \#(f^{-1}(y) \cap F1) = 2\}$, and the set of cusp values $C2 = f(C1)$ are discrete sets. Clearly, $N = R2 \cup F2 \cup FF2 \cup C2$. Let $S_{2,y}$ be the connected component of $R1$, $F1$, $FF1$ or $C1$ which contains y. Then $\mathscr{S}_2 := \{S_{2,y} : y \in N\}$.

Let $RR3 = R1 \cap f^{-1}(R2)$, $RF3 = R1 \cap f^{-1}(F2)$, $RFF3 = R1 \cap f^{-1}(FF2)$, and $RC3 = R1 \cap f^{-1}(C2)$. Then these are regular points of f at which f takes regular, fold, double fold and cusp values, respectively. Let $F3 = F1 \cap f^{-1}(F2)$ and $FF3 = F1 \cap f^{-1}(FF2)$. Then these are fold points at which the values of f are folds and double folds, respectively. Let $C3 = C1$. $RR3$ is open in M, $RF3$ and $F3$ are embedded one-dimensional submanifolds of M, and $RFF3$, $RC3$, $FF3$ and $C3$ are discrete sets. Let $S_{3,x}$ be the connected component of $RR3$, $RF3$, $RFF3$, $RC3$, $F3$, $FF3$ or $C3$ which contains x. Then $\mathscr{S}_3 := \{S_{3,x} : x \in M\}$.

By a *stratification* of M (or N), we mean a partition $\mathscr{S}$ of M into connected embedded submanifolds, called *strata*, satisfying the following two conditions:

(a) $\mathscr{S}$ is locally finite, i.e., each point of M has a neighborhood intersecting only finitely many strata,

(b) (Axiom of the Frontier) if U and V are strata and $\bar{U} \cap V \neq \emptyset$, then $V \subset \bar{U}$.

Note that $\mathscr{S}_1, \mathscr{S}_2$ and $\mathscr{S}_3$ are stratifications.

Next we study the behavior of f on $\mathscr{S}_1, \mathscr{S}_2$, and $\mathscr{S}_3$. First note that if $S \in \mathscr{S}_1$ then $f|S$ is an immersion. If $T \in \mathscr{S}_3$, then T is a submanifold of some $S \in \mathscr{S}_1$, and hence $f|T$ is an immersion.

Choose any $T \in \mathscr{S}_3(f)$ and let $S \in \mathscr{S}_2(f)$ be the stratum containing $f(T)$. Since T is a connected component of $f^{-1}(S)$, $f|T: T \to S$ is proper and hence a covering map (in fact, if X is Hausdorff and Y is connected and compactly generated and if $h: X \to Y$ is a proper local homeomorphism, then h is a covering map onto Y; see Theorem 4.2 of Palais [15] or Lemma 2.1 of Wilson [20]).

We define a *stratification map* $\bar{f}$ from a stratification $\mathscr{S}$ to a stratification $\mathscr{T}$ to be a function from the set $\mathscr{S}$ to the set $\mathscr{T}$ satisfying the following conditions: $\bar{f}$ preserves the dimension of strata; if $S \in \mathscr{S}$, then $\bar{f}(\operatorname{star}(S)) \subseteq \operatorname{star}(\bar{f}(S))$, where $\operatorname{star} S = \{T \in \mathscr{S} : \bar{T} \cap S \neq \emptyset\}$. We call $\bar{f}$ an *equivalence* if there is a stratification map $\bar{g}$ from $\mathscr{T}$ to $\mathscr{S}$ such that $\bar{f} \circ \bar{g}$ and $\bar{g} \circ \bar{f}$ are the identity set maps.

The stable map f induces a stratification map $\bar{f}: \mathscr{S}_3 \to \mathscr{S}_2$ in the obvious way.

If $\mathscr{S}$ and $\mathscr{T}$ are stratifications of M or N and each $S \in \mathscr{S}$ is contained in some $T \in \mathscr{T}$, then $\mathscr{S}$ is a *substratification* of $\mathscr{T}$. We say $\mathscr{S}$ is a *simple substratification* of $\mathscr{T}$ if, in addition, each $S \in \mathscr{S}$ is simply connected. We say $\mathscr{S}$ is *local* if, for each $x \in M$ (or N), each neighborhood U of x contains a neighborhood V of x such that $V \cap S$ is connected (possibly empty) for every $S \in \mathscr{S}$.

If $\mathscr{S}_4$ is a substratification of $\mathscr{S}_2(f)$, then f induces a substratification $\mathscr{S}_5(f)$ of $\mathscr{S}_3(f)$ as follows:

$$\mathscr{S}_5(f) = \{\text{connected components of } f^{-1}(S) : S \in \mathscr{S}_4\} .$$

Note that, for $T \in \mathscr{S}_5(f)$ and S the stratum in $\mathscr{S}_4$ containing $f(T)$, $f|T: T \to S$ is a covering space. If S is simply connected, then T is simply connected, and in fact f maps T diffeomorphically onto S. So if $\mathscr{S}_4$ is a simple substratification of $\mathscr{S}_2(f)$, then $\mathscr{S}_5(f)$ is a simple substratification of $\mathscr{S}_3(f)$, and f maps each stratum $S \in \mathscr{S}_5(f)$ diffeomorphically onto $\bar{f}(S) \in \mathscr{S}_4$. Suppose $\mathscr{S}_4$ is also local; then $\mathscr{S}_5(f)$ is local and, in addition, if S, T and U are in $\mathscr{S}_5(f)$ with S and T in $\bar{U}$, then $f(S) \neq f(T)$.

Theorem 2.1. *Suppose M and N are two-dimensional manifolds and $f, g \in C^\infty(M, N)$ are proper, stable and equivalent, that is, $k \circ f = g \circ h$ for certain diffeomorphisms $h \in C^\infty(M, M)$ and $k \in C^\infty(N, N)$.*

(1) *Then h induces equivalences $\bar{h}: \mathscr{S}_1(f) \to \mathscr{S}_1(g)$ and $\bar{h}: \mathscr{S}_3(f) \to \mathscr{S}_3(g)$, k induces an equivalence $\bar{k}: \mathscr{S}_2(f) \to \mathscr{S}_2(g)$, $\bar{k} \circ \bar{f} = \overline{k \circ f} = \overline{g \circ h} = \bar{g} \circ \bar{h}$.*

(2) *If $\mathscr{S}_4$ is a (simple) substratification of $\mathscr{S}_2(g)$, then $k \circ f$ and g induce (simple) substratifications $\mathscr{S}_5(k \circ f)$ of $\mathscr{S}_3(f)$ and $\mathscr{S}_5(g)$ of $\mathscr{S}_3(g)$, and h induces an equivalence $h': \mathscr{S}_5(h \circ f) \to \mathscr{S}_5(g)$ such that $g(h'(S)) = k \circ f(S)$ for all $S \in \mathscr{S}_5(k \circ f)$.*

The proof of this theorem is straightforward.

3. Examples

It is natural to ask to what extent the stable map f is determined by the induced map $\bar{f}: \mathscr{S}_3(f) \to \mathscr{S}_2(f)$. Theorem 4.1 answers this question; but first we look at three examples which illustrate the precautions one has to take in trying to formulate a converse to Theorem 2.1. In Example 3.1 we see two stable maps f and g which induce equivalent stratification maps (i.e., there are equivalences $\bar{h}: \mathscr{S}_3(f) \to \mathscr{S}_3(g)$ and $\bar{k}: \mathscr{S}_2(f) \to \mathscr{S}_2(g)$ such that $\bar{k} \circ \bar{f} = \bar{g} \circ \bar{h}$) but are not themselves equivalent. The difficulty arises because there is a stratum $S \in \mathscr{S}_3(f)$ on which f is not a diffeomorphism but rather a double covering. In Examples 3.2 and 3.3 we see that, even when all the strata are simply connected, $\mathscr{S}_3(f)$ can equal $\mathscr{S}_3(g)$ (in Example 3.2) or $\mathscr{S}_2(f)$ can equal $\mathscr{S}_2(g)$ (in Example 3.3) without f and g being equivalent.

Example 3.1. Let f and g be stable maps from S^2 to $\boldsymbol{R}^2$ having $\mathscr{S}_i(f) = \mathscr{S}_i(g) = \mathscr{S}_i$ for $i = 1, 2$ and 3, where $\mathscr{S}_1, \mathscr{S}_2$ and $\mathscr{S}_3$ are shown in Figures 1, 2 and 3 respectively. The capital letters in these figures indicate 0-strata.

We require that the induced maps $\bar{f}, \bar{g}: \mathscr{S}_3 \to \mathscr{S}_2$ be equal, and map each 0-stratum in $\mathscr{S}_3$ to the 0-stratum in $\mathscr{S}_2$ bearing the same letter. The action of $\bar{f}$ and $\bar{g}$ on 1-strata and 2-strata can be deduced by its action on 0-strata.

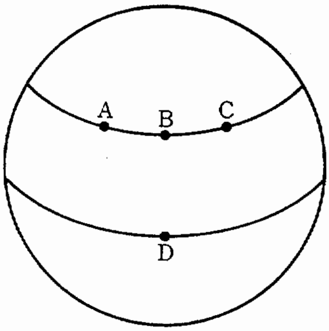


Fig. 1

$\mathscr{S}_1$: $A, B, C, D \in C1$, the line segments are in $F1$, open strata are in $R1$.

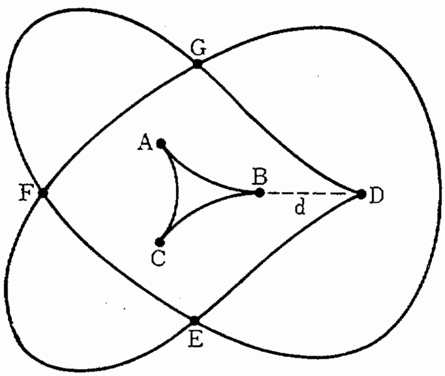


Fig. 2

$\mathscr{S}_2$: open strata are in $R2$, line segments are in $F2$, $A, B, C, D \in C2$, $E, F, G \in FF2$, (the dotted line d is not a stratum).

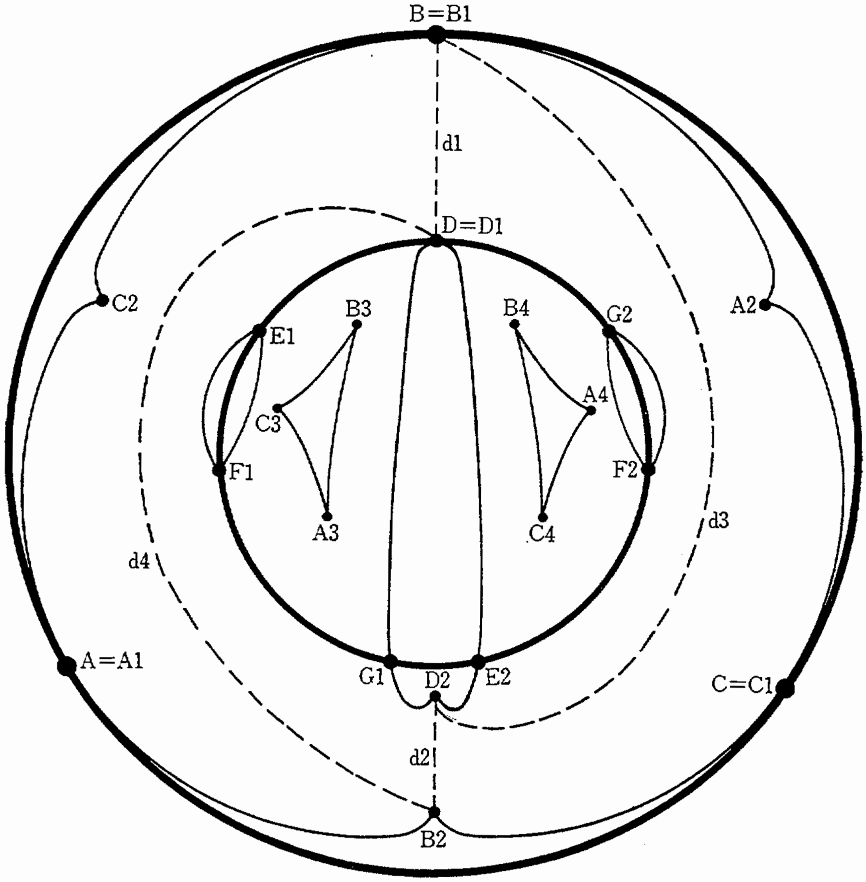


Fig. 3

The northern polar cap of S^2 is a single $\mathcal{S}_3$ stratum. We show here only the complement of the northern polar cap.

$\mathcal{S}_3$: open strata are in $RR3$; heavy-lined 1-strata are in $F3$, fine-lined 1-strata are in $RF3$; $E1, E2, F1, F2, G1, G2 \in FF3$, $A1, B1, C1 \in C3$, $A2, A3, A4, B2, B3, B4, C2, C3, C4 \in RC3$, $RFF3 = \emptyset$, (the dotted lines $d1, d2, d3$, and $d4$ are not $\mathcal{S}_3$ strata).

Notice that there are nonsimply connected 2-strata (annuli) in $\mathcal{S}_3$ and $\mathcal{S}_2$; the restrictions of f and g to the source annulus must be double covering maps (one can see this by counting the 1-strata in the boundaries of the source and target annuli). So it is possible to choose f and g such that $f^{-1}(d) = d1 \cup d2$ and $g^{-1}(d) = d3 \cup d4$.

If we form a simple substratification $\mathcal{S}_4$ by adjoining d to $\mathcal{S}_2$, then $\mathcal{S}_5(f)$ is gotten by adjoining $d1$ and $d2$ to $\mathcal{S}_3$, whereas $\mathcal{S}_5(g)$ is gotten by adjoining $d3$

and $d4$ to $\mathscr{S}_3$. Note that $\mathscr{S}_5(f)$ and $\mathscr{S}_5(g)$ are not equivalent (for example, an equivalence would have to map $d1$, which is bounded by two cusp points, to either $d3$ or $d4$, each of which is bounded by a cusp point and a regular point, and this is impossible) and so, by Theorem 2.1, f and g are not equivalent.

Example 3.2. Let f and g be stable maps from $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ into $\boldsymbol{R}^2$ formed by first projecting S^2 orthogonally onto the unit disk D^2 in the xy-plane, and then immersing the disk into $\boldsymbol{R}^2$ as shown in Fig. 4. Both f and g have $\partial D = \{(x, y, 0) : x^2 + y^2 = 1\} \subset S^2$ as their only fold curve. The left side of Fig. 4 shows $\mathscr{S}_3(f)$ and $\mathscr{S}_3(g)$ for the disk (these are lifted by the projection $S^2 \to D^2$ to get $\mathscr{S}_3(f)$ and $\mathscr{S}_3(g)$ on S^2).

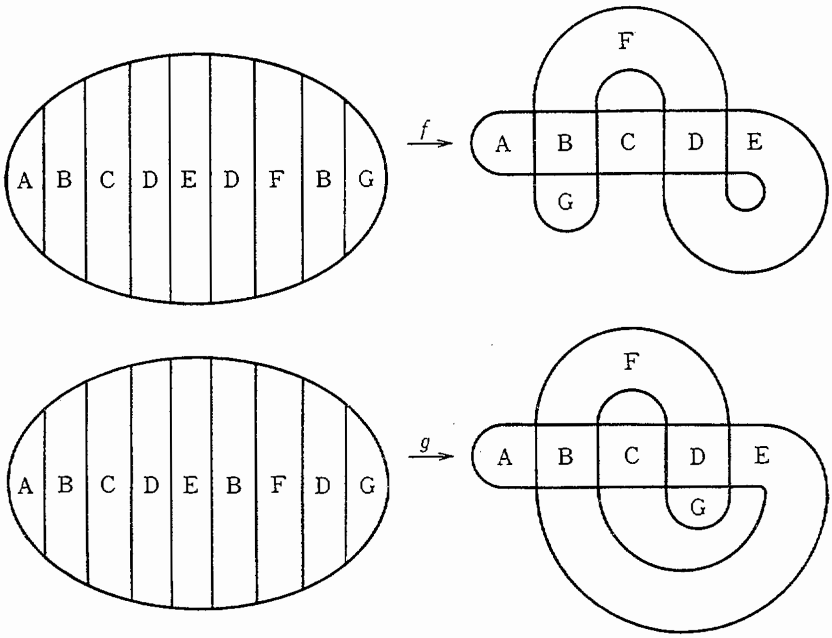
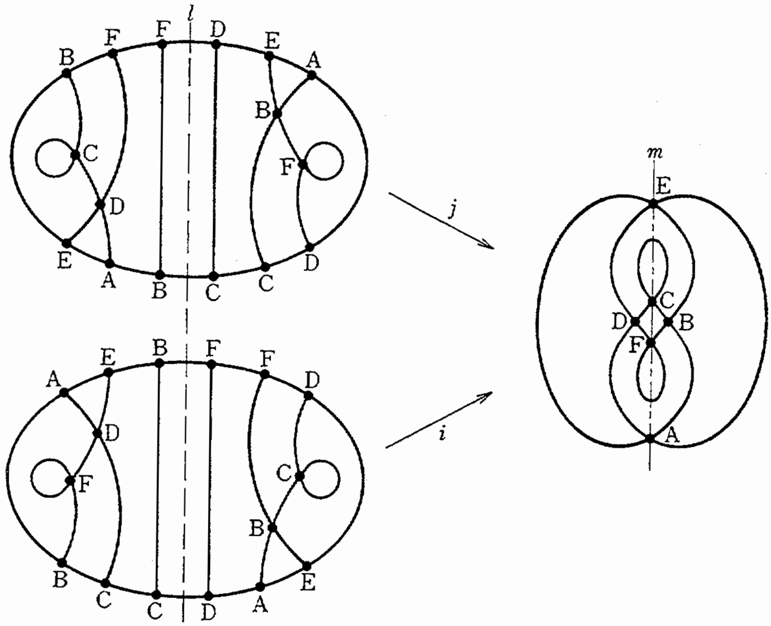


Fig. 4

Note that $\mathscr{S}_2(f)$ and $\mathscr{S}_2(g)$ are not equivalent (for example, $\mathscr{S}_2(f)$ has a 2-stratum, the unbounded component of $\boldsymbol{R}^2 - f(S^2)$, with five 1-strata in its boundary, whereas $\mathscr{S}_2(g)$ has no such 2-stratum). Hence, by Theorem 2.1, f and g are not equivalent.

Example 3.3. We now construct stable maps f, g and h from $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ into $\boldsymbol{R}^2$, each having $\partial D^2 = \{x^2 + y^2 = 1, z = 0\}$ as its only fold curve. First consider Milnor's well-known example (Fig. 5) of two immersions i and j from D^2 into $\boldsymbol{R}^2$. The stratifications $\mathscr{S}_1$, $\mathscr{S}_2$ and $\mathscr{S}_3$ can be defined for immersions of D^2 into $\boldsymbol{R}^2$ which satisfy condition WII (see Proposition 1.3), where ∂D^2 is considered as the fold curve of the immersion. The right side of Fig. 5 shows $\mathscr{S}_2(i) = \mathscr{S}_2(j)$ with the $FF2$-points marked; the left hand side shows $\mathscr{S}_3(i)$ and $\mathscr{S}_3(j)$ with the $FF3$-points (on the boundary) and $RFF3$-points (in the interior) marked.



(The dotted lines l and m are not strata.)

Fig. 5

Letting π_N be the projection of the northern hemisphere H_N onto D, and π_S the projection of the southern hemisphere H_S onto D, we define f by $f|H_N = i \circ \pi_N$ and $f|H_S = i \circ \pi_S$, g by $g|H_N = i \circ \pi_N$ and $g|H_S = j \circ \pi_S$, and h by $h|H_N = j \circ \pi_N$ and $h|H_S = j \circ \pi_S$, where we identify H_N and H_S so that corresponding letters along the equator match.

Note that f is equivalent to h by means of reflecting $\boldsymbol{R}^2$ about the line m and S^2 about the verticle plane in $\boldsymbol{R}^3$ containing the line l (see Fig. 5).

However, $\mathscr{S}_3(f)$ is not equivalent to $\mathscr{S}_3(g)$. Note that in both $\mathscr{S}_3(f)$ and $\mathscr{S}_3(g)$ there are four 2-strata having the property that each has only a single 1-stratum in its boundary. An equivalence of $\mathscr{S}_3(f)$ and $\mathscr{S}_3(g)$ would have to preserve such 2-strata. But in $\mathscr{S}_3(f)$ there is a path consisting of two 1-strata and three 0-strata which connects two of these 2-strata; there is no such path in $\mathscr{S}_3(g)$, as there would have to be if $\mathscr{S}_3(f)$ were equivalent to $\mathscr{S}_3(g)$. Hence, by Theorem 2.1, f and g are not equivalent.

4. Equivalence of stratification-equivalent stable maps

Theorem 4.1. *Suppose M and N are two-dimensional manifolds, and $f, g \in C^\infty(M,N)$ are proper and stable. Suppose there is a diffeomorphism $k \in C^\infty(N,N)$ which induces an equivalence $\bar{k}: \mathscr{S}_2(f) \to \mathscr{S}_2(g)$. Suppose $\mathscr{S}_4$ is a local simple substratification of $\mathscr{S}_2(g)$, and $\mathscr{S}_5(g)$ and $\mathscr{S}_5(k\circ f)$ are the induced substratifications of $\mathscr{S}_3(g)$ and $\mathscr{S}_3(f)$. Assume further that there is an equivalence $\bar{h}: \mathscr{S}_5(k\circ f) \to \mathscr{S}_5(g)$ such that if $S \in \mathscr{S}_5(k\circ f)$ consists of fold (cusp) points, then $\bar{h}(S)$ consists of fold (cusp) points and such that $k\circ f(S) = g(\bar{h}(S))$ for all $S \in \mathscr{S}_5(k\circ f)$. Then there is a unique C^∞ diffeomorphism $h: M \to M$ such that $k\circ f = g\circ h$.*

Remark. Since a compact subset of N intersects only finitely many strata of $\mathscr{S}_2(g)$, it is easy to see that $\mathscr{S}_2(g)$ necessarily has a local simple substratification $\mathscr{S}_4$.

Proof. Define h by letting $h\,|\,S$ be $(g\,|\,\bar{h}(S))^{-1}\circ k\circ(f\,|\,S)$ for each $S \in \mathscr{S}_5(k\circ f)$. Clearly h is well-defined and bijective. We need only to check that h is C^∞ and nonsingular in a neighborhood of each fold and cusp point of f. The hypothesis that $\mathscr{S}_4$ is local guarantees that h is continuous.

Let p be a fold point of f; then $q = h(p)$ is a fold point of g. By (1.1.a), C^∞ coordinate systems (V,φ) and (W,ψ) can be chosen around q and $g(q)$ respectively such that for $G: \boldsymbol{R}^2 \to \boldsymbol{R}^2$ given by $G(x,y) = (x^2, y)$ we have $(\psi\circ g\circ\varphi^{-1})(x,y) = G(x,y)$ in a neighborhood of $(0,0)$. Let $S_1 = \{(x,y) \in \boldsymbol{R}^2 : x < 0\}$, $S_2 = \{(x,y) \in \boldsymbol{R}^2 : x > 0\}$ and $S_3 = \{(x,y) \in \boldsymbol{R}^2 : x = 0\}$. Note that $\mathscr{S}_1(G) = \mathscr{S}_2(G) = \mathscr{S}_3(G) = \{S_1, S_2, S_3\}$; also note that $\bar{G}: \mathscr{S}_3(G) \to \mathscr{S}_2(G)$ is given by $\bar{G}(S_1) = \bar{G}(S_2) = S_2$ and $\bar{G}(S_3) = S_3$. In what follows, we will frequently restrict G (and other maps) to a neighborhood, say U, of $(0,0)$, in which case we will denote $G\,|\,U$ by G; also we will denote $S_1 \cap U$ by S_1, etc.

By (1.1.a), a coordinate system (U,α) can be chosen about p such that $\mathscr{S}_1(F) = \mathscr{S}_2(F) = \mathscr{S}_3(F) = \{S_1, S_2, S_3\}$ for $F = \psi\circ k\circ f\circ\alpha^{-1}$. $\bar{F}: \mathscr{S}_3(F) \to \mathscr{S}_2(F)$ is given by $\bar{F}(S_1) = \bar{F}(S_2) = S_2$ and $\bar{F}(S_3) = S_3$, and $\bar{\varphi}\circ\bar{h}\circ\overline{\alpha^{-1}}: \mathscr{S}_3(F) \to \mathscr{S}_3(G)$ takes S_i to S_i for $i = 1, 2, 3$. The following commutative diagram summarizes the mappings we have defined:

$$
\begin{tikzcd}
\alpha(U) \arrow[ddrrr, "{F=(u(x,y),v(x,y))}"] & & & \\
U \subseteq M \arrow[u, "\alpha"] \arrow[dr, "k\cdot f"] \arrow[dd, "h"] & & & \\
 & N \supseteq W \arrow[rr, "\psi"] & & \boldsymbol{R}^2 \\
V \subseteq M \arrow[ur, "g"'] \arrow[d, "\varphi"] & & & \\
\boldsymbol{R}^2 \arrow[uurrr, "{G=(x^2,y)}"'] & & &
\end{tikzcd}
$$

Then, for $i = 1, 2$ and 3, $(\varphi \circ h \circ \alpha^{-1}) | S_i = (G | S_i)^{-1} \circ (F | S_i)$. Thus

$$\varphi \circ h \circ \alpha^{-1} = \begin{cases} (u(x,y)^{1/2}, v(x,y)) , & x \geq 0 , \\ (-u(x,y)^{1/2}, v(x,y)) , & x \leq 0 . \end{cases}$$

In Proposition 4.2 we show that $\varphi \circ h \circ \alpha^{-1}$ is C^∞ and nonsingular at $(0, 0)$, and hence that h is C^∞ and nonsingular at p, as required for the proof of Theorem 4.1.

Proposition 4.2. *With notation as above,*

$$\varphi \circ h \circ \alpha^{-1} = \begin{cases} (u(x,y)^{1/2}, v(x,y)) , & x \geq 0 , \\ (-u(x,y)^{1/2}, v(x,y)) , & x \leq 0 \end{cases}$$

is C^∞ and nonsingular at $(0, 0)$.

Proof. We use the notation λ_x, λ_y for the partial derivatives of a differentiable function $\lambda: \boldsymbol{R}^2 \to \boldsymbol{R}$. Let J denote the Jacobian matrix of $F = (u(x, y), v(x, y))$. Recall that a point p is a good singular point of F if

$$\begin{aligned} &\det J = u_x v_y - u_y v_x = 0 \text{ at } p, \text{ and} \\ (4.3) \qquad &\operatorname{grad}(\det J) = (u_{xx}v_y + u_x v_{xy} - u_{xy}v_x - u_y v_{xx} , \\ &\qquad u_{xy}v_y + u_x v_{yy} - u_{yy}v_x - u_y v_{xy}) \neq 0 \text{ at } p . \end{aligned}$$

Note that $\det J = 0$ implies that the rank of J is < 2, and that $\operatorname{grad}(\det J) \neq 0$ implies that the rank of $J > 0$. Hence

$$(4.4) \qquad \text{the rank of } J = 1 \text{ at a good point} .$$

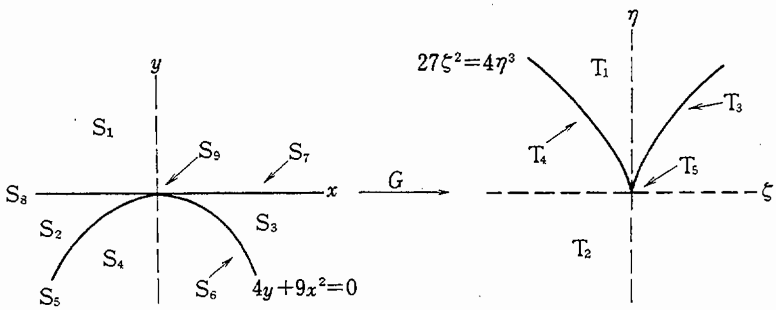
By assumption $u(0, y) = 0$ for all y, hence $u_y(0, y) = 0$, $u_{yy}(0, y) = 0$, etc. for all y. Since the restriction of F to the y-axis is an immersion (by the definition of a fold point), the vector $(u_y(0, y), v_y(0, y)) \neq 0$, that is, $v_y(0, y) \neq 0$ for all y. Since fold points are good points, (4.4) implies that $u_x(0, y) = 0$ for all y, and (4.3) implies that $u_{xx}(0, y) \neq 0$ for all y.

By Taylor's Theorem, there is a C^∞ function $w(x, y)$ such that $u(x, y) = u(0, y) + u_x(0, y)x + w(x, y)x^2 = w(x, y)x^2$. Note that $w(0, y) = \frac{1}{2}u_{xx}(0, y) \neq 0$ for all y.

Hence, in some neighborhood of the y-axis, $w(x, y) > 0$ and $\varphi \circ h \circ \alpha^{-1}(x, y) = (x(w(x, y))^{1/2}, v(x, y))$ and is C^∞. Since $\dfrac{\partial}{\partial x}(x(w(x, y))^{1/2})(0, y) = w(0, y)^{1/2} \neq 0$, $\dfrac{\partial}{\partial y}(v(x, y))(0, y) = v_y(0, y) \neq 0$, $\varphi \circ h \circ \alpha^{-1}$ is nonsingular at $(0, y)$ for all y.

Our next goal is to prove that the mapping h is C^∞ and nonsingular at cusp points of f.

The mapping $G: \boldsymbol{R}^2 \to \boldsymbol{R}^2$ given by $G(x, y) = (xy + 2x^3, y + 3x^2)$ is proper and stable, with a cusp point at $(0, 0)$ and fold points elsewhere on the x-axis. The image of the x-axis in the $\zeta\eta$-plane is $\{(\zeta, \eta): 27\zeta^2 = 4\eta^3\}$. The inverse image of this set under G is the union of the x-axis and the curve $\{(x, y): 4y + 9x^2 = 0\}$. The stratifications defined by G, as shown in Fig. 6, are: $\mathscr{S}_1(G) = \{S_1 \cup \cdots \cup S_6, S_7, S_8, S_9\}$; $\mathscr{S}_3(G) = \{S_1, \cdots, S_9\}$, where $S_1, \cdots, S_4 \subset RR3$, S_5 and $S_6 \subset RF3$, S_7 and $S_8 \subset F3$ and $S_9 \subset C3$; $\mathscr{S}_2(G) = \{T_1, \cdots, T_5\}$, where T_1 and $T_2 \subset R2$, T_3 and $T_4 \subset F2$, and $T_5 = C2$. The induced map $\bar{G}: \mathscr{S}_3(G) \to \mathscr{S}_2(G)$ maps S_1, S_2 and S_3 to T_1, S_4 to T_2, S_6 and S_8 to T_4, S_5 and S_7 to T_3, and S_9 to T_5.



Solid lines indicate strata.

Fig. 6

Let p be a cusp point of f; then $q = h(p)$ is a cusp point of g. By (1.1.b), coordinate systems (V, φ) and (W, ψ) can be chosen around q and $g(q)$ respectively such that, for $G: \boldsymbol{R}^2 \to \boldsymbol{R}^2$ given by $G(x, y) = (xy + 2x^3, y + 3x^2) = (xy - x^3, y) \circ H$, where H is the diffeomorphism $(x, y + 3x^2)$, we have $(\psi \circ g \circ \varphi^{-1})(x, y) = G(x, y)$ in a neighborhood of $(0, 0)$.

Again by (1.1.b) we get a coordinate system (U, α) about p such that for $F = \psi \circ k \circ f \circ \alpha^{-1}$ we have $\mathscr{S}_3(F) = \{S_1, \cdots, S_9\}$ and $\bar{\varphi} \circ \bar{h} \circ \overline{\alpha^{-1}}: \mathscr{S}_3(F) \to \mathscr{S}_3(G)$ takes S_i to S_i for $i = 1, \cdots, 9$. To complete the proof of Theorem 4.1, we need to show that h is C^∞ and nonsingular at p, that is, $\varphi \circ h \circ \alpha^{-1}$ is C^∞ and nonsingular at $(0, 0)$. For $i = 1, \cdots, 9$, $(\varphi \circ h \circ \alpha^{-1}) | S_i = (G | S_i)^{-1} \circ (F | S_i)$.

Proposition 4.5. *With the same notation as above, the map H defined by $H | S_i = (G | S_i)^{-1} \circ (F | S_i)$, for $i = 1, \cdots, 9$, is C^∞ and nonsingular at* $(0, 0)$.

Proof. Let J denote the Jacobian matrix of F, and the singular set of F, which is the x-axis, be parameterized by x. Then the definition of a cusp point states that

$$(4.6) \qquad (u_x(0, 0), v_x(0, 0)) = 0\ , \qquad (u_{xx}(0, 0), v_{xx}(0, 0)) \neq 0\ .$$

Since fold and cusp points are good points, (4.4) implies that J has rank one on the x-axis. Since F is an immersion at fold points, the image of $J(x, 0)$ at fold points, and hence also at $(0, 0)$ by the continuity of J, is tangent to the curve $27\zeta^2 = 4\eta^3$. Hence $J(0, 0)$ maps the xy-plane onto the η-axis. Therefore $u_y(0, 0) = 0$ and $v_y(0, 0) \neq 0$. If $v_y(0, 0)$ were negative, then $v(0, y)$ would be negative for sufficiently small $y > 0$, that is, $(0, y)$ would be in S_1 and $F(0, y) = (u, v)(0, y)$ would be in T_2, contradicting that $\bar{F}(S_1) = T_1$. Thus

$$u_x(0, 0) = u_y(0, 0) = v_x(0, 0) = 0 , \qquad v_y(0, 0) > 0 . \tag{4.7}$$

By (4.7) and (4.3), $\operatorname{grad}(\det J) = (u_{xx}v_y, u_{xy}v_y) \neq 0$ at $(0, 0)$. Using the fact that the gradient vector of a function is always perpendicular to the level surface of the function, that is, $\operatorname{grad}(\det J)(0, 0)$ is perpendicular to the set $\det J = 0$, which is the x-axis, and using that $\det J(0, y) > 0$ if $y > 0$ and $\det J(0, y) < 0$ if $y < 0$, we see that

$$u_{xx}(0, 0) = 0 , \qquad u_{xy}(0, 0) > 0 . \tag{4.8}$$

There are C^∞ functions $a(x, y)$, $b(x)$, $c(y)$, $d(x, y)$ and $e(x)$ such that

$$\begin{aligned} u(x, y) &= a(x, y)xy + b(x)x^3 + c(y)y^2 , \\ v(x, y) &= d(x, y)y + e(x)x^2 . \end{aligned} \tag{4.9}$$

Indeed, by Taylor's theorem, $u(x, 0) = b(x)x^3$, $u(0, y) = d(y)y^2$. Letting $w(x, y) = u(x, y) - b(x)x^3 - c(y)y^2$, we see that $w(x, 0) = w(0, y) = 0$ for all x and y, and so $w(x, y) = a(x, y)xy$. Similarly, $v(x, 0) = e(x)x^2$, and letting $z(x, y) = v(x, y) - e(x)x^2$ and noting that $z(x, 0) = 0$ for all x we have $z(x, y) = d(x, y)y$.

By assumption, $27u(x, 0)^2 = 4v(x, 0)^3$. Thus, by (4.9), $27(b(x)x^3)^2 = 4(e(x)x^2)^3$, which implies that

$$27b(x)^2 = 4e(x)^3 \qquad \text{for all } x . \tag{4.10}$$

Together, (4.6), (4.7), (4.8), (4.10) and the fact that $u(x, 0) \geq 0$ for $x \geq 0$ imply that

$$a(0, 0), b(0), d(0, 0) \text{ and } e(0) \text{ are positive} . \tag{4.11}$$

We say that two maps f and g are *right-equivalent at* 0 if there is a C^∞ map h, nonsingular at 0, such that $h(0) = 0$ and $f = g \circ h$ near 0.

Note that in the derivation of (4.9) and (4.11), the precise positions of the strata S_5 and S_6 were not used, but only that they lie below the x-axis. So (4.9) and (4.11) also describe the form of any map which is right-equivalent at 0 to F by a local diffeomorphism which leaves strata S_1, S_7 and S_8 invariant.

Now we replace F by maps F_i, right-equivalent to F at 0, such that $G - F_i$ becomes successively simpler.

Since 0 is a nondegerate critical point of u, by the Morse lemma, u is right-equivalent at 0 to $f(x, y) = x^2 - y^2$ and so its zero set contains a manifold transverse to the x-axis at 0. Thus there is a C^∞ function f such that $f(0) = 0$ and $u(f(y), y) = 0$ near 0. Let $H_1(x, y) = (x + f(y), y)$ and $F_1 = F \circ H_1$. Then

$$\begin{aligned} F_1(x, y) &= (u_1(x, y), v_1(x, y)) \\ &= (a_1(x, y)xy + b_1(x)x^3 + c_1(y)y^2, d_1(x, y)y + e_1(x)x^2) \end{aligned}$$

for some C^∞ functions a_1, b_1, c_1, d_1 and e_1 with $a_1(0, 0), b_1(0), d_1(0, 0)$ and $e_1(0)$ positive. Now $c_1(y)y^2 = u(f(y), y) = 0$ near 0, so $c_1(y) = 0$ near 0.

Since $v_1(0, y) = d_1(0, y)y$, $h_1(y) = v_1(0, y)^{-1}$ is C^∞ near 0 with $h_1(0) = 0$ and $h_1'(0) > 0$. Let $H_2(x, y) = (x, h_1(y))$ and $F_2 = F_1 \circ H_2$. Then

$$\begin{aligned} F_2(x, y) &= (u_2(x, y), v_2(x, y)) \\ &= (a_2(x, y)xy + b_2(x)x^3 + c_2(y)y^2, d_2(x, y)y + e_2(x)x^2) \end{aligned}$$

for some C^∞ functions a_2, b_2, c_2, d_2 and e_2 with $a_2(0, 0), b_2(0), d_2(0, 0)$ and $e_2(0)$ positive. Note that $v_2(0, y) = v_1(0, h_1(y)) = y$ near 0, so $d_2(0, y) = 1$ near 0. Since H_2 leaves the y-axis invariant, $c_2(y) = 0$ near 0.

Since $u_2(x, 0) = b_2(x)x^3$, $h_2(x) = ((\frac{1}{2}u_2(x, 0))^{1/3})^{-1}$ is C^∞ near 0 with $h_2(0) = 0$ and $h_2'(0) > 0$. Let $H_3(x, y) = (h_2(x), y)$ and $F_3 = F_2 \circ H_3$. Then $F_3 = (u_3, v_3) = (a_3xy + b_3x^3 + c_3y^2, d_3y + e_3x^2)$ for some C^∞ functions a_3, b_3, c_3, d_3 and e_3 with a_3, b_3, d_3 and e_3 positive at 0. Now $u_3(x, 0) = 2x^3$, i.e., $b_3(x) = 2$ near 0; hence $e_3(x) = 3$ near 0 by (4.10). Since H_3 leaves the y-axis pointwise invariant, $d_3(0, y) = 1$ and $c_3(y) = 0$ near 0.

Since $F_3(x, 0) = G(x, 0)$ near 0, the vectors $DF_3(x, 0) \cdot \partial/\partial y = (a_3(x, 0)x, d_3(x, 0))$ and $DG(x, 0) \cdot \partial/\partial y = (x, 1)$, being tangent to the curve $27\zeta^2 = 4\eta^3$ and having the same base point, are parallel for each x near 0, i.e., $a_3(x, 0) = d_3(x, 0)$ for x near 0. Let $H_4(x, y) = (x, y/d_3(x, y))$ and $F_4 = F_3 \circ H_4$. Then $F_4 = (a_4xy + b_4x^3 + c_4y^2, d_4y + e_4x^2)$ for some C^∞ functions a_4, b_4, c_4, d_4 and e_4 with a_3, b_3, d_3 and e_3 positive at 0. Since H_4 leaves the x- and y-axes pointwise invariant, $b_4(x) = 2$, $c_4(y) = 0$, $d_4(0, y) = 1$ and $e_4(x) = 3$ near 0. Furthermore, it is easy to see that $a_4(x, 0) = d_4(x, 0) = 1$ near 0. Thus $G - F_4 = (a_5xy^2, d_5xy^2)$ for some C^∞ functions a_5 and d_5.

Proposition 4.5 is now an immediate consequence of the following corollary of Tougeron's generalized implicit function theorem [16, Proposition II.1.1]: if $\varphi, \varphi^1: (\boldsymbol{R}^p, 0) \to (\boldsymbol{R}^q, 0)$ are C^∞, with $q \leq p$, and if the component functions of $\varphi - \varphi^1$ are contained in $MI(\varphi)^2$, where $I(\varphi)$ is the ideal in $C^\infty(\boldsymbol{R}^p)$ generated by determinants of the $q \times q$ minors of the Jacobian matrix of φ and M is the maximal ideal in $C^\infty(\boldsymbol{R}^p)$ of functions which vanish at 0, then there is a C^∞ mapping $f: (\boldsymbol{R}^p, 0) \to (\boldsymbol{R}^p, 0)$ with component functions in $MI(\varphi)$ such that $\varphi \circ (\mathrm{Id} + f) = \varphi^1$ near 0. Note that $I(G) = yC^\infty(\boldsymbol{R}^2)$ and so the hypotheses are satisfied for $\varphi = G$ and $\varphi^1 = F_4$. Since the resulting f has component functions in yM, Id $+ f$ is nonsingular at 0. Thus G is right-equivalent at 0 to F_4 and hence to F.

This concludes the proof of Proposition 4.5 and hence of Theorem 4.1.

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